Deductive Program Verification with WHY3

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http://why3.lri.fr/ejcp-2022

ÉJCP 2022

Software is hard. — DONALD KNUTH

Ensure the absence of bugs

Several approaches exist: model checking, abstract interpretation, etc.

In this lecture: deductive verification

1. provide a program with a specification: a mathematical model
2. build a formal proof showing that the code respects the specification

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1. provide a program with a specification: a mathematical model
2. build a formal proof showing that the code respects the specification

First proof of a program: Alan Turing, 1949

```
u := 1
for r = 0 to n - 1 do
  v := u
  for s = 1 to r do
    u := u + v
```

Ensure the absence of bugs

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First theoretical foundation: Floyd-Hoare logic, 1969

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First grand success in practice: metro line 14, 1998

tool: Atelier B, proof by refinement

Some other major success stories

- Flight control software in A380, 2005

  safety proof: the absence of execution errors
  tool: Astrée, abstract interpretation

  proof of functional properties
  tool: Caveat, deductive verification

- Hyper-V — a native hypervisor, 2008

  tools: VCC + automated prover Z3, deductive verification

- CompCert — verified C compiler, 2009

  tool: Coq, generation of the correct-by-construction code

- seL4 — an OS micro-kernel, 2009

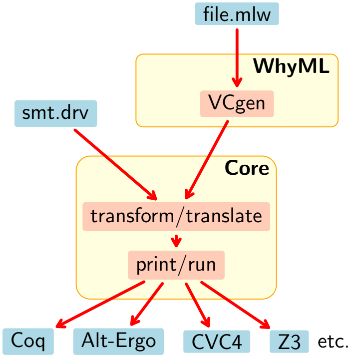
  tool: Isabelle/HOL, deductive verification

- CakeML — verified ML compiler, 2016

  tool: HOL4, deductive verification, self-bootstrap

1. Tool of the day

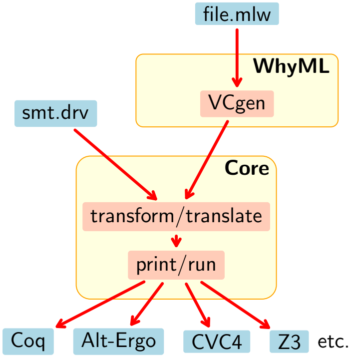
WHY3 in a nutshell



WHY3 in a nutshell

WHYML, a programming language

- type polymorphism • variants
- limited support for higher order
- pattern matching • exceptions
- `break`, `continue`, and `return`
- ghost code and ghost data (CAV 2014)
- mutable data with controlled aliasing
- contracts • loop and type invariants



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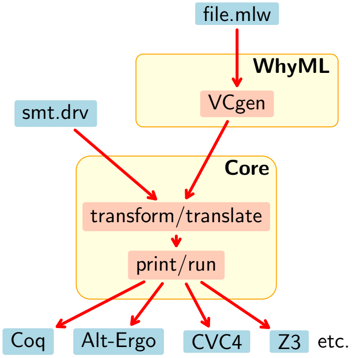
WHYML, a specification language

- polymorphic & algebraic types
- limited support for higher order
- inductive predicates

(FroCos 2011) (CADE 2013)



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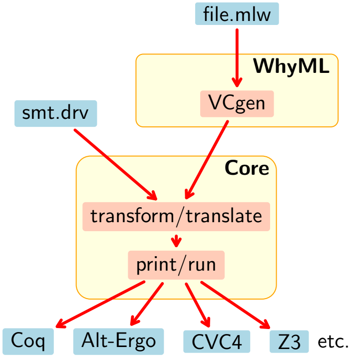

WHY3, a program verification tool

- VC generation using WP or fast WP
- 70+ VC *transformations* (≈ tactics)
- support for 25+ ATP and ITP systems

(Boogie 2011) (ESOP 2013) (VSTTE 2013)

Three different ways of using WHY3

- as a logical language
  - a convenient front-end to many theorem provers
- as a programming language to prove algorithms
  - see examples in our gallery
    `http://toccata.lri.fr/gallery/why3.en.html`
- as an intermediate verification language
  - Java programs: Krakatoa (Marché Paulin Urbain)
  - C programs: Frama-C (Marché Moy)
  - Ada programs: SPARK 2014 (Adacore)
  - probabilistic programs: EasyCrypt (Barthe et al.)

Example: maximum subarray problem

```
let maximum_subarray (a: array int): int
  ensures { forall l h: int. 0 <= l <= h <= length a -> sum a l h <= result }
  ensures { exists l h: int. 0 <= l <= h <= length a /\ sum a l h  = result }
```

Kadane's algorithm

```
(*  | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  *)
(*  ......|######## max ########|..............                     *)
(*  ..............................|### cur ####                     *)

let maximum_subarray (a: array int): int
  ensures { forall l h: int. 0 <= l <= h <= length a -> sum a l h <= result }
  ensures { exists l h: int. 0 <= l <= h <= length a /\ sum a l h  = result }
=
  let ref max = 0 in
  let ref cur = 0 in



  for i = 0 to length a - 1 do




    cur += a[i];
    if cur < 0 then       cur <- 0;
    if cur > max then       max <- cur
  done;
  max
```

```
(*  | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  *)
(*  ......|######## max ########|.............                       *)
(*  .............................|### cur ####                       *)

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  ensures { forall l h: int. 0 <= l <= h <= length a -> sum a l h <= result }
  ensures { exists l h: int. 0 <= l <= h <= length a /\ sum a l h  = result }
=
  let ref max = 0 in
  let ref cur = 0 in
  let ghost ref cl = 0 in


  for i = 0 to length a - 1 do
    invariant { forall l: int. 0 <= l <= i -> sum a l i <= cur }
    invariant { 0 <= cl <= i /\ sum a cl i = cur }


    cur += a[i];
    if cur < 0 then begin cur <- 0; cl <- i+1 end;
    if cur > max then       max <- cur
  done;
  max
```

Kadane's algorithm

```
(*  | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |  *)
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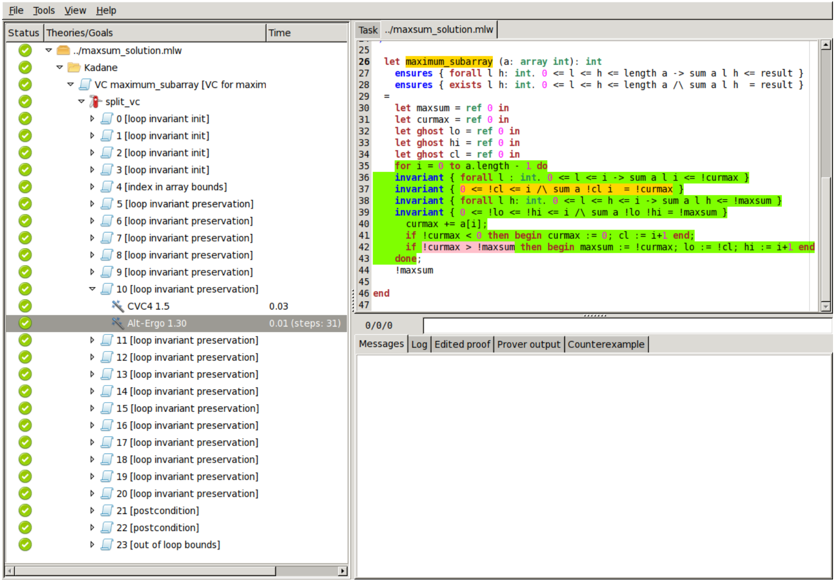
=
  let ref max = 0 in
  let ref cur = 0 in
  let ghost ref cl = 0 in
  let ghost ref lo = 0 in
  let ghost ref hi = 0 in
  for i = 0 to length a - 1 do
    invariant { forall l: int. 0 <= l <= i -> sum a l i <= cur }
    invariant { 0 <= cl <= i /\ sum a cl i = cur }
    invariant { forall l h: int. 0 <= l <= h <= i -> sum a l h <= max }
    invariant { 0 <= lo <= hi <= i /\ sum a lo hi = max }
    cur += a[i];
    if cur < 0 then begin cur <- 0; cl <- i+1 end;
    if cur > max then begin max <- cur; lo <- cl; hi <- i+1 end
  done;
  max
```

Why3 proof session



2. Program correctness

Pure terms

t	$::=$	$\ldots, -1, 0, 1, \ldots, 42, \ldots$	integer constants
	$\mid$	$\texttt{true} \mid \texttt{false}$	Boolean constants
	$\mid$	$u \mid v \mid w$	immutable variable
	$\mid$	$x \mid y \mid z$	dereferenced pointer
	$\mid$	$t\ op\ t$	binary operation
	$\mid$	$op\ t$	unary operation
op	$::=$	$+ \mid - \mid *$	arithmetic operations
	$\mid$	$= \mid \neq \mid < \mid > \mid \leqslant \mid \geqslant$	arithmetic comparisons
	$\mid$	$\wedge \mid \vee \mid \neg$	Boolean connectives

- two data types: mathematical integers and Booleans
- well-typed terms evaluate without errors (no division)
- evaluation of a term does not change the program memory

Program expressions

e	$::=$	`skip`	do nothing
	$\mid$	t	pure term
	$\mid$	$x \leftarrow t$	assignment
	$\mid$	e ; e	sequence
	$\mid$	`let` $v = e$ `in` e	binding
	$\mid$	`let ref` $x = e$ `in` e	allocation
	$\mid$	`if` t `then` e `else` e	conditional
	$\mid$	`while` t `do` e `done`	loop

- three types: integers, Booleans, and `unit`
- references (pointers) are not first-class values
- expressions can allocate and modify memory
- well-typed expressions evaluate without errors

Typed expressions

$$
\begin{array}{lcl}
\texttt{skip} & : & \texttt{unit} \\
t_\tau & : & \tau \\
x_\tau \leftarrow t_\tau & : & \texttt{unit} \\
e_{\texttt{unit}} \texttt{ ; } e_\varsigma & : & \varsigma \\
\texttt{let } v_\tau = e_\tau \texttt{ in } e_\varsigma & : & \varsigma \\
\texttt{let ref } x_\tau = e_\tau \texttt{ in } e_\varsigma & : & \varsigma \\
\texttt{if } t_{\texttt{bool}} \texttt{ then } e_\varsigma \texttt{ else } e_\varsigma & : & \varsigma \\
\texttt{while } t_{\texttt{bool}} \texttt{ do } e_{\texttt{unit}} \texttt{ done} & : & \texttt{unit}
\end{array}
$$

- $\tau ::= \texttt{int} \mid \texttt{bool}$ and $\varsigma ::= \tau \mid \texttt{unit}$
- references (pointers) are not first-class values
- expressions can allocate and modify memory
- well-typed expressions evaluate without errors

$$
\begin{aligned}
x \leftarrow e \quad &\equiv \quad \texttt{let}\ v = e\ \texttt{in}\ x \leftarrow v \\
\texttt{if}\ e\ \texttt{then}\ e_1\ \texttt{else}\ e_2 \quad &\equiv \quad \texttt{let}\ v = e\ \texttt{in}\ \texttt{if}\ v\ \texttt{then}\ e_1\ \texttt{else}\ e_2 \\
\texttt{if}\ e_1\ \texttt{then}\ e_2 \quad &\equiv \quad \texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ \texttt{skip} \\
e_1\ \&\&\ e_2 \quad &\equiv \quad \texttt{if}\ e_1\ \texttt{then}\ e_2\ \texttt{else}\ \texttt{false} \\
e_1\ ||\ e_2 \quad &\equiv \quad \texttt{if}\ e_1\ \texttt{then}\ \texttt{true}\ \texttt{else}\ e_2
\end{aligned}
$$

Example

```
let ref sum = 1 in
let ref count = 0 in
while sum ⩽ n do
  count ← count + 1;
  sum ← sum + 2 * count + 1
done;
count
```

What is the result of this expression for a given n?

Example — *ISQRT*

```
let ref sum = 1 in
let ref count = 0 in
while sum ⩽ n do
  count ← count + 1;
  sum ← sum + 2 ∗ count + 1
done;
count
```

What is the result of this expression for a given `n`?

Informal specification:

- at the end, `count` contains the truncated square root of `n`
- for instance, given $n = 42$, the returned value is 6

Hoare triples

A statement about program correctness:

$$\{P\}\ e\ \{Q\}$$

- P precondition property
- e expression
- Q postcondition property

What is the meaning of a Hoare triple?

$\{P\}\ e\ \{Q\}$ if we execute e in a state that satisfies P,
then the computation either diverges
or terminates in a state that satisfies Q

This is partial correctness: we say nothing about termination.

Examples of valid Hoare triples for partial correctness:

- $\{x = 1\}\ x \leftarrow x + 2\ \{x = 3\}$
- $\{x = y\}\ x + y\ \{\mathtt{result} = 2y\}$
- $\{\exists v.\, x = 4v\}\ x + 42\ \{\exists w.\, \mathtt{result} = 2w\}$
- $\{\mathtt{true}\}$ `while true do skip done` $\{\boxed{\boxed{\mathtt{false}}}\}$

Examples

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  - after this loop, *everything* is trivially verified
  - ergo: not proving termination can be fatal

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In our square root example:

$$\{?\}\ \mathit{ISQRT}\ \{?\}$$

Examples

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In our square root example:

$$\{n \geqslant 0\}\ \mathit{ISQRT}\ \{?\}$$

Examples

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  - after this loop, *everything* is trivially verified
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In our square root example:

$$\{n \geqslant 0\}\ \mathit{ISQRT}\ \{\texttt{result}^2 \leqslant n < (\texttt{result} + 1)^2\}$$

3. Weakest precondition calculus

How can we establish the correctness of a program?

One solution: Edsger Dijkstra, 1975

Predicate transformer $\mathrm{WP}(e, Q)$

- e expression
- Q postcondition

computes the weakest precondition P such that $\{P\}\ e\ \{Q\}$

Intuition of WP

$$x \leftarrow 3 * x * y \qquad \{\ x \text{ is even}\ \}$$

$\{ \text{ } 3xy \text{ is even } \} \qquad x \leftarrow 3 * x * y \qquad \{ \text{ } x \text{ is even } \}$

Intuition of WP

$\{\ 3xy \text{ is even}\ \}$ $\quad x \leftarrow 3 * x * y \quad$ $\{\ x \text{ is even}\ \}$

$\{\ Q[s]\ \}$ $\quad x \leftarrow s \quad$ $\{\ Q[x]\ \}$

Intuition of WP

$\{ \ 3xy \text{ is even} \ \}$ $\quad x \leftarrow 3 * x * y \quad$ $\{ \ x \text{ is even} \ \}$

$\{ \ Q[s] \ \}$ $\quad x \leftarrow s \quad$ $\{ \ Q[x] \ \}$

if c then $\quad e_1 \quad$ $\{ \ Q \ \}$
else $\quad e_2$

$\{ \; 3xy \text{ is even} \; \} \qquad x \leftarrow 3 * x * y \qquad \{ \; x \text{ is even} \; \}$

$\{ \; Q[s] \; \} \qquad x \leftarrow s \qquad \{ \; Q[x] \; \}$

`if` c `then` $\quad e_1 \; Q \quad \{ \; Q \; \}$

`else` $\quad e_2 \; Q$

Intuition of WP

$\{\ 3xy \text{ is even}\ \}$ $\quad x \leftarrow 3 * x * y \quad$ $\{\ x \text{ is even}\ \}$

$\{\ Q[s]\ \}$ $\quad x \leftarrow s \quad$ $\{\ Q[x]\ \}$

```
if c then P1 e1 Q
     else P2 e2 Q
```

$\{\ Q\ \}$

Intuition of WP

$\{ \text{ } 3xy \text{ is even } \}$ $\quad x \leftarrow 3 * x * y \quad$ $\{ \text{ } x \text{ is even } \}$

$\{ \text{ } Q[s] \text{ } \}$ $\quad x \leftarrow s \quad$ $\{ \text{ } Q[x] \text{ } \}$

$\{ \text{ if } c \text{ then } P_1 \text{ else } P_2 \text{ } \}$ $\quad \texttt{if } c \texttt{ then } P_1 \, e_1 \, Q \texttt{ else } P_2 \, e_2 \, Q \quad$ $\{ \text{ } Q \text{ } \}$

$\{\ 3xy \text{ is even}\ \}$	$x \leftarrow 3 * x * y$	$\{\ x \text{ is even}\ \}$
$\{\ Q[s]\ \}$	$x \leftarrow s$	$\{\ Q[x]\ \}$
$\{\ \text{if } c \text{ then } P_1$ $\text{else } P_2\ \}$	`if` c `then` $P_1\ e_1\ Q$ `else` $P_2\ e_2\ Q$	$\{\ Q\ \}$
	`if` c `then` e	$\{\ Q\ \}$

Intuition of WP

$\{\ 3xy \text{ is even}\ \}$ $\quad x \leftarrow 3 * x * y \quad$ $\{\ x \text{ is even}\ \}$

$\{\ Q[s]\ \}$ $\quad x \leftarrow s \quad$ $\{\ Q[x]\ \}$

$\{\ \text{if } c \text{ then } P_1$
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`else` $P_2\ e_2\ Q$ $\quad$ $\{\ Q\ \}$

`if` c `then` $P\ e\ Q$ $\quad$ $\{\ Q\ \}$

Intuition of WP

$\{\ 3xy \text{ is even}\ \}$	$x \leftarrow 3 * x * y$	$\{\ x \text{ is even}\ \}$
$\{\ Q[s]\ \}$	$x \leftarrow s$	$\{\ Q[x]\ \}$
$\{\ \text{if } c \text{ then } P_1$ $\text{else } P_2\ \}$	$\texttt{if}\ c\ \texttt{then}\ P_1\, e_1\, Q$ $\texttt{else}\ P_2\, e_2\, Q$	$\{\ Q\ \}$
$\{\ \text{if } c \text{ then } P$ $\text{else } Q\ \}$	$\texttt{if}\ c\ \texttt{then}\ P\, e\, Q$	$\{\ Q\ \}$

Intuition of WP

$\{\ 3xy \text{ is even}\ \}$ $\quad x \leftarrow 3 * x * y \quad$ $\{\ x \text{ is even}\ \}$

$\{\ Q[s]\ \}$ $\quad x \leftarrow s \quad$ $\{\ Q[x]\ \}$

$\{\ \text{if } c \text{ then } P_1$
$\quad \text{else } P_2\ \}$ $\quad$ **if** c **then** $P_1\ e_1\ Q$
else $P_2\ e_2\ Q$ $\quad$ $\{\ Q\ \}$

$\{\ \text{if } c \text{ then } P$
$\quad \text{else } Q\ \}$ $\quad$ **if** c **then** $P\ e\ Q$ $\quad$ $\{\ Q\ \}$

while c **do** e **done** $\quad$ $\{\ Q\ \}$

Intuition of WP

$\{\ 3xy$ is even $\}$	$x \leftarrow 3 * x * y$	$\{\ x$ is even $\}$
$\{\ Q[s]\ \}$	$x \leftarrow s$	$\{\ Q[x]\ \}$
$\{$ if c then P_1 else P_2 $\}$	`if` c `then` $P_1\ e_1\ Q$ `else` $P_2\ e_2\ Q$	$\{\ Q\ \}$
$\{$ if c then P else Q $\}$	`if` c `then` $P\ e\ Q$	$\{\ Q\ \}$
?	`while` c `do` e `done`	$\{\ Q\ \}$

$$
\begin{array}{rcl}
\mathrm{WP}(\texttt{skip}, Q) & \equiv & Q \\
\mathrm{WP}(t, Q) & \equiv & Q[\texttt{result} \mapsto t] \\
\mathrm{WP}(x \leftarrow t, Q) & \equiv & Q[x \mapsto t] \\
\mathrm{WP}(e_1 \texttt{ ; } e_2, Q) & \equiv & \mathrm{WP}(e_1, \mathrm{WP}(e_2, Q)) \\
\mathrm{WP}(\texttt{let } v = e_1 \texttt{ in } e_2, Q) & \equiv & \mathrm{WP}(e_1, \mathrm{WP}(e_2, Q)[v \mapsto \texttt{result}]) \\
\mathrm{WP}(\texttt{let ref } x = e_1 \texttt{ in } e_2, Q) & \equiv & \mathrm{WP}(e_1, \mathrm{WP}(e_2, Q)[x \mapsto \texttt{result}]) \\
\mathrm{WP}(\texttt{if } t \texttt{ then } e_1 \texttt{ else } e_2, Q) & \equiv & \quad (t \rightarrow \mathrm{WP}(e_1, Q)) \land \\
 & & (\neg t \rightarrow \mathrm{WP}(e_2, Q))
\end{array}
$$

```
if odd q then r ← r + p;
p ← p + p;
q ← half q
```

```
if odd q then
    r ← r + p
else
    skip;
p ← p + p;
q ← half q
```

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if odd q then
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```

$Q[p, q, r]$

```
  if odd q then
      r ← r + p
  else
      skip;
  p ← p + p;
Q[p, half q, r]
  q ← half q
Q[p, q, r]
```

```
if odd q then
    r ← r + p
else
    skip;
Q[p + p, half q, r]
  p ← p + p;
Q[p, half q, r]
  q ← half q
Q[p, q, r]
```

```
if odd q then

    r ← r + p
  Q[p + p, half q, r]
else

    skip;
  Q[p + p, half q, r]
p ← p + p;
Q[p, half q, r]
q ← half q
Q[p, q, r]
```

```
if odd q then
  Q[p + p, half q, r + p]
    r ← r + p
  Q[p + p, half q, r]
else
  Q[p + p, half q, r]
    skip;
  Q[p + p, half q, r]
 p ← p + p;
Q[p, half q, r]
 q ← half q
Q[p, q, r]
```

$$(\texttt{odd}\ q \rightarrow Q[p+p, \texttt{half}\ q, r+p]) \land$$
$$(\neg\ \texttt{odd}\ q \rightarrow Q[p+p, \texttt{half}\ q, r])$$

```
if odd q then
```

$Q[p+p, \texttt{half}\ q, r+p]$

$r \leftarrow r + p$

$Q[p+p, \texttt{half}\ q, r]$

```
else
```

$Q[p+p, \texttt{half}\ q, r]$

```
skip;
```

$Q[p+p, \texttt{half}\ q, r]$

$p \leftarrow p + p$;

$Q[p, \texttt{half}\ q, r]$

$q \leftarrow \texttt{half}\ q$

$Q[p, q, r]$

Definition of WP: loops

$\mathrm{WP}(\texttt{while}\ t\ \texttt{do}\ e\ \texttt{done}, Q) \equiv$	
$\exists J : \texttt{Prop}.$	some *invariant property* J
$J \wedge$	that holds at the loop entry
$\forall x_1 \ldots x_k.$	and is preserved
$(J \wedge t \rightarrow \mathrm{WP}(e, J)) \wedge$	after a single iteration,
$(J \wedge \neg t \rightarrow Q)$	is strong enough to prove Q

$x_1 \ldots x_k$ references modified in e

We cannot know the values of the modified references after n iterations

- therefore, we prove preservation and the post for arbitrary values
- the invariant must provide all the needed information about the state

Definition of WP: annotated loops

Finding an appropriate invariant is difficult in the general case

- this is equivalent to constructing a proof of Q by induction

We can ease the task of automated tools by providing annotations:

$\mathrm{WP}(\texttt{while } t \texttt{ invariant } J \texttt{ do } e \texttt{ done}, Q) \equiv$	the given invariant J
$J \wedge$	holds at the loop entry,
$\forall x_1 \dots x_k.$	is preserved after
$(J \wedge t \rightarrow \mathrm{WP}(e, J)) \wedge$	a single iteration,
$(J \wedge \neg t \rightarrow Q)$	and suffices to prove Q

$x_1 \dots x_k$ references modified in e

Russian Peasant Multiplication

```
let ref p = a in
let ref q = b in
let ref r = 0 in
while q > 0 invariant J[p, q, r] do
    if odd q then r ← r + p;
    p ← p + p;
    q ← half q
done;
r
```

result $= a * b$

```
let ref p = a in
let ref q = b in
let ref r = 0 in
while q > 0 invariant J[p, q, r] do
    if odd q then r ← r + p;
    p ← p + p;
    q ← half q
done;
r = a * b
r
```

```
let ref p = a in
let ref q = b in
let ref r = 0 in
while q > 0 invariant J[p,q,r] do
    if odd q then r ← r + p;
    p ← p + p;
    q ← half q
  J[p, q, r]
done;
r = a ∗ b
r
```

```
let ref p = a in
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let ref r = 0 in
while q > 0 invariant J[p,q,r] do
    (odd q → J[p + p, half q, r + p]) ∧
  (¬ odd q → J[p + p, half q, r])
    if odd q then r ← r + p;
    p ← p + p;
    q ← half q
  J[p, q, r]
done;
r = a ∗ b
r
```

```
let ref p = a in
let ref q = b in
let ref r = 0 in
J[p,q,r] ∧
∀p q r. J[p,q,r] →
  (q > 0 →
      (odd q → J[p + p, half q, r + p]) ∧
    (¬ odd q → J[p + p, half q, r])) ∧
  (q ⩽ 0 →
    r = a ∗ b)
while q > 0 invariant J[p,q,r] do
    if odd q then r ← r + p;
    p ← p + p;
    q ← half q
done;
r
```

$$J[a, b, 0] \land$$
$$\forall p\, q\, r.\ J[p, q, r] \to$$
$$(q > 0 \to$$
$$(\texttt{odd}\ q \to J[p + p, \texttt{half}\ q, r + p]) \land$$
$$(\neg\, \texttt{odd}\ q \to J[p + p, \texttt{half}\ q, r])) \land$$
$$(q \leqslant 0 \to$$
$$r = a * b)$$

```
let ref p = a in
let ref q = b in
let ref r = 0 in
while q > 0 invariant J[p,q,r] do
    if odd q then r ← r + p;
    p ← p + p;
    q ← half q
done;
r
```

Theorem

For any e and Q, the triple $\{\mathrm{WP}(e, Q)\}\ e\ \{Q\}$ is valid.

Can be proved by induction on the structure of the program e
w.r.t. some reasonable semantics (axiomatic, operational, etc.)

Corollary

To show that $\{P\}\ e\ \{Q\}$ is valid, it suffices to prove $P \rightarrow \mathrm{WP}(e, Q)$.

This is what WHY3 does.

4. Run-time safety

Run-time errors

Some operations can fail if their safety preconditions are not met:

- arithmetic operations: division par zero, overflows, etc.
- memory access: `NULL` pointers, buffer overruns, etc.
- assertions

Run-time errors

Some operations can fail if their safety preconditions are not met:

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- memory access: `NULL` pointers, buffer overruns, etc.
- assertions

A correct program must not fail:

$\{P\}\ e\ \{Q\}$ if we execute e in a state that satisfies P,
then there will be no run-time errors
and the computation either diverges
or terminates normally in a state that satisfies Q

Assertions

A new kind of expression:

$$e \quad ::= \quad \ldots$$
$$\quad | \quad \texttt{assert}\ R \qquad \text{fail if } R \text{ does not hold}$$

The corresponding weakest precondition rule:

$$\mathrm{WP}(\texttt{assert}\ R, Q) \quad \equiv \quad R \wedge Q \quad \equiv \quad R \wedge (R \rightarrow Q)$$

The second version is useful in practical deductive verification.

Unsafe operations

We could add other partially defined operations to the language:

e	$::=$	$\dots$	
	$\mid$	$t \ \mathtt{div}\ t$	Euclidean division
	$\mid$	$a\,[\,t\,]$	array access
	$\mid$	$\dots$	

and define the WP rules for them:

$$\mathrm{WP}(t_1\ \mathtt{div}\ t_2, Q) \equiv t_2 \neq 0 \land Q[\mathtt{result} \mapsto (t_1\ \mathtt{div}\ t_2)]$$

$$\mathrm{WP}(a[t], Q) \equiv 0 \leqslant t < |a| \land Q[\mathtt{result} \mapsto a[t]]$$

$$\dots$$

But we would rather let the programmers do it themselves.

5. Functions and contracts

Subroutines

We may want to delegate some functionality to functions:

$$\texttt{let}\ f\ (v_1 : \tau_1)\ \ldots\ (v_n : \tau_n) : \varsigma\ \mathscr{C} = e \qquad \text{defined function}$$

$$\texttt{val}\ f\ (v_1 : \tau_1)\ \ldots\ (v_n : \tau_n) : \varsigma\ \mathscr{C} \qquad \text{abstract function}$$

Function behaviour is specified with a contract:

$$\begin{array}{llll} \mathscr{C} ::= & \texttt{requires}\ P & & \text{precondition} \\ & \texttt{writes}\ x_1 \ldots x_k & & \text{modified global references} \\ & \texttt{ensures}\ Q & & \text{postcondition} \end{array}$$

Postcondition Q may refer to the initial value of a global reference: x°

```
let incr_r (v: int): int writes r
  ensures result = r° ∧ r = r° + v
= let u = r in r ← u + v ; u
```

Subroutines

We may want to delegate some functionality to functions:

$\texttt{let}\ f\ (v_1 : \tau_1) \ldots (v_n : \tau_n) : \varsigma\ \mathscr{C} = e$	defined function
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Function behaviour is specified with a contract:

$\mathscr{C}$ ::=	$\texttt{requires}\ P$	precondition
	$\texttt{writes}\ x_1 \ldots x_k$	modified global references
	$\texttt{ensures}\ Q$	postcondition

Postcondition Q may refer to the initial value of a global reference: x°

Verification condition ($\vec{x}$ are all global references mentioned in f):

$$\mathrm{VC}(\texttt{let}\ f \ldots) \quad \equiv \quad \forall \vec{x}\, \vec{v}.\, P \rightarrow \mathrm{WP}(e, Q)[\vec{x}^\circ \mapsto \vec{x}]$$

One more expression:

$$e \ ::= \ \dots \ \mid \ f \ t \ \dots \ t \qquad \text{function call}$$

and its weakest precondition rule:

$$\mathrm{WP}(f \ t_1 \ \dots \ t_n, Q) \ \equiv \ P_f[\vec{v} \mapsto \vec{t}] \ \wedge$$
$$(\forall \vec{\boldsymbol{x}} \, \forall \mathtt{result}. \, Q_f[\vec{v} \mapsto \vec{t}, \vec{x}^\circ \mapsto \vec{w}] \rightarrow Q)[\vec{w} \mapsto \vec{x}]$$

P_f	precondition of f	$\vec{\boldsymbol{x}}$	references modified in f
Q_f	postcondition of f	$\vec{x}$	references used in f
$\vec{v}$	formal parameters of f	$\vec{w}$	fresh variables

Modular proof: when verifying a function call, we only use the function's contract, not its code.

Examples

```
let max (x y: int) : int
  ensures  { result >= x /\ result >= y }
  ensures  { result =  x \/ result =  y }
= if x >= y then x else y
```

```
val ref r : int  (* declare a global reference *)

let incr_r (v: int) : int
  requires { v > 0 }
  writes   { r }
  ensures  { result = old r /\ r = old r + v }
= let u = r in
  r <- u + v;
  u
```

6. Total correctness: termination

Problem: prove that the program terminates for every initial state that satisfies the precondition.

It suffices to show that

- every loop makes a finite number of iterations
- recursive function calls cannot go on indefinitely

Solution: prove that every loop iteration and every recursive call decreases a certain value, called variant, with respect to some well-founded order.

For example, for signed integers, a practical well-founded order is

$$i \prec j \quad = \quad i < j \land 0 \leqslant j$$

A new annotation:

$$e \;::=\; \dots \;\mid\; \texttt{while}\ t\ \texttt{invariant}\ J\ \texttt{variant}\ t \cdot \prec\ \texttt{do}\ e\ \texttt{done}$$

The corresponding weakest precondition rule:

$$\begin{array}{l}
\mathrm{WP}(\texttt{while}\ t\ \texttt{invariant}\ J\ \texttt{variant}\ s \cdot \prec\ \texttt{do}\ e\ \texttt{done}, Q) \equiv \\
\quad J \wedge \\
\quad \forall x_1 \dots x_k. \\
\quad\quad (J \wedge t \rightarrow \mathrm{WP}(e, J \wedge s \prec w)[w \mapsto s]) \wedge \\
\quad\quad (J \wedge \neg t \rightarrow Q)
\end{array}$$

$x_1 \dots x_k$ references modified in e

w a fresh variable (the variant at the start of the iteration)

Termination of recursive functions

A new contract clause:

$$
\begin{array}{l}
\texttt{let rec } f\ (v_1 : \tau_1)\ \ldots\ (v_n : \tau_n)\ :\ \varsigma \\
\qquad\qquad \texttt{requires } P_f \\
\qquad\qquad \texttt{variant } s \cdot \prec \\
\qquad\qquad \texttt{writes } \vec{\boldsymbol{x}} \\
\qquad\qquad \texttt{ensures } Q_f \\
\qquad = e
\end{array}
$$

For each recursive call of f in e:

$$
\begin{aligned}
\mathrm{WP}(f\ \ t_1\ \ldots\ t_n, Q) \ \equiv\ & P_f[\vec{v} \mapsto \vec{t}\,] \wedge s[\vec{v} \mapsto \vec{t}\,] \prec s[\vec{x} \mapsto \vec{x}^\circ] \wedge \\
& (\forall \vec{\boldsymbol{x}}\, \forall \texttt{result}.\, Q_f[\vec{v} \mapsto \vec{t}, \vec{x}^\circ \mapsto \vec{w}] \to Q)[\vec{w} \mapsto \vec{x}]
\end{aligned}
$$

$s[\vec{v} \mapsto \vec{t}\,]$	variant at the call site	$\vec{x}$	references used in f
$s[\vec{x} \mapsto \vec{x}^\circ]$	variant at the start of f	$\vec{w}$	fresh variables

Mutually recursive functions must have

- their own variant terms
- a common well-founded order

Thus, if f calls $g\ t_1\ \ldots\ t_n$, the variant decrease precondition is

$$s_g[\vec{v}_g \mapsto \vec{t}\,] \prec s_f[\vec{x} \mapsto \vec{x}^\circ]$$

$\vec{v}_g$	formal parameters of g
$s_g[\vec{v}_g \mapsto \vec{t}\,]$	variant of g at the call site
$s_f[\vec{x} \mapsto \vec{x}^\circ]$	variant of f at the start of f

7. Exceptions

Exceptions as destinations

Execution of a program can lead to

- divergence — the computation never ends
  - total correctness ensures against non-termination

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  - each potential exception E gets its own postcondition Q_E

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  - partial correctness ensures conformance to the contract
- exceptional termination — produces *a different kind of result*
  - the contract should also cover exceptional termination
  - each potential exception E gets its own postcondition Q_E
  - partial correctness: *if* E *is raised, then* Q_E *holds*

Exceptions as destinations

Execution of a program can lead to

- divergence — the computation never ends
  - total correctness ensures against non-termination
- abnormal termination — the computation fails
  - partial correctness ensures against run-time errors
- normal termination — the computation produces a result
  - partial correctness ensures conformance to the contract
- exceptional termination — produces *a different kind of result*

```
exception Not_found

val binary_search (a: array int) (v: int) : int
  requires { forall i j. 0 ⩽ i ⩽ j < length a → a[i] ⩽ a[j] }
  ensures  { 0 ⩽ result < length a ∧ a[result] = v }
  raises   { Not_found → forall i. 0 ⩽ i < length a → a[i] ≠ v }
```

Our language keeps growing:

$$
\begin{array}{rcll}
e & ::= & \dots & \\
  & | & \texttt{raise}\ \mathsf{E} & \text{raise an exception} \\
  & | & \texttt{try}\ e\ \texttt{with}\ \mathsf{E} \rightarrow e & \text{catch an exception}
\end{array}
$$

WP handles two postconditions now:

$$\mathrm{WP}(\texttt{skip}, Q, Q_{\mathsf{E}}) \quad \equiv \quad Q$$

Just another semicolon

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e	$::=$	$\dots$	
	$\mid$	`raise` E	raise an exception
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WP handles two postconditions now:

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\mathrm{WP}(\texttt{skip}, Q, Q_{\mathsf{E}}) & \equiv & Q \\
\mathrm{WP}(\texttt{raise}\ \mathsf{E}, Q, Q_{\mathsf{E}}) & \equiv & Q_{\mathsf{E}} \\
\mathrm{WP}(e_1\ ;\ e_2, Q, Q_{\mathsf{E}}) & \equiv & \mathrm{WP}(e_1, \mathrm{WP}(e_2, Q, Q_{\mathsf{E}}), Q_{\mathsf{E}})
\end{array}
$$

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$$
\begin{array}{rcll}
e & ::= & \dots & \\
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\end{array}
$$

Just another let-in

Exceptions can carry data:

$$
\begin{array}{rcll}
e & ::= & \dots & \\
 & | & \texttt{raise}\ \mathsf{E}\ t & \text{raise an exception} \\
 & | & \texttt{try}\ e\ \texttt{with}\ \mathsf{E}\ v \rightarrow e & \text{catch an exception}
\end{array}
$$

Still, all needed mechanisms are already in WP:

$$
\begin{array}{rcl}
\mathrm{WP}(t, Q, Q_{\mathsf{E}}) & \equiv & Q[\texttt{result} \mapsto t] \\
\mathrm{WP}(\texttt{raise}\ \mathsf{E}\ t, Q, Q_{\mathsf{E}}) & \equiv & Q_{\mathsf{E}}[\texttt{result} \mapsto t] \\
\mathrm{WP}(\texttt{let}\ v = e_1\ \texttt{in}\ e_2, Q, Q_{\mathsf{E}}) & \equiv & \\
 & & \mathrm{WP}(e_1, \mathrm{WP}(e_2, Q, Q_{\mathsf{E}})[v \mapsto \texttt{result}], Q_{\mathsf{E}}) \\
\mathrm{WP}(\texttt{try}\ e_1\ \texttt{with}\ \mathsf{E}\ v \rightarrow e_2, Q, Q_{\mathsf{E}}) & \equiv & \\
 & & \mathrm{WP}(e_1, Q, \mathrm{WP}(e_2, Q, Q_{\mathsf{E}})[v \mapsto \texttt{result}])
\end{array}
$$

A new contract clause:

$$\begin{array}{l} \texttt{let}\ f\ (v_1 : \tau_1)\ \ldots\ (v_n : \tau_n)\ :\ \varsigma \\ \qquad \texttt{requires}\ P_f \\ \qquad \texttt{writes}\ \vec{\boldsymbol{x}} \\ \qquad \texttt{ensures}\ Q_f \\ \qquad \texttt{raises}\ \mathsf{E} \to Q_{\mathsf{E}f} \\ \quad = e \end{array}$$

Verification condition for the function definition:

$$\mathrm{VC}(\texttt{let}\ f\ \ldots) \quad \equiv \quad \forall \vec{x}\,\vec{v}.\, P_f \to \mathrm{WP}(e, Q_f, Q_{\mathsf{E}f})[\vec{x}^{\circ} \mapsto \vec{x}]$$

Weakest precondition rule for the function call:

$$\begin{array}{rcl} \mathrm{WP}(f\ t_1\ \ldots\ t_n, Q, Q_{\mathsf{E}}) & \equiv & P_f[\vec{v} \mapsto \vec{t}]\ \wedge \\ & & (\forall \vec{\boldsymbol{x}}\, \forall \texttt{result}.\, Q_f[\vec{v} \mapsto \vec{t}, \vec{x}^{\circ} \mapsto \vec{w}] \to Q)[\vec{w} \mapsto \vec{x}]\ \wedge \\ & & (\forall \vec{\boldsymbol{x}}\, \forall \texttt{result}.\, Q_{\mathsf{E}f}[\vec{v} \mapsto \vec{t}, \vec{x}^{\circ} \mapsto \vec{w}] \to Q_{\mathsf{E}})[\vec{w} \mapsto \vec{x}] \end{array}$$

8. Ghost code

Compute a Fibonacci number using a recursive function in $O(n)$:

```
let rec aux (a b n: int): int
  requires { 0 <= n }
  requires {                                                    }
  ensures  {
                                                                }
  variant  { n }
= if n = 0 then a else aux b (a+b) (n-1)

let fib_rec (n: int): int
  requires { 0 <= n }
  ensures  { result = fib n }
= aux 0 1 n

(* fib_rec 5 = aux 0 1 5 = aux 1 1 4 = aux 1 2 3 =
               aux 2 3 2 = aux 3 5 1 = aux 5 8 0 = 5 *)
```

Compute a Fibonacci number using a recursive function in $O(n)$:

```
let rec aux (a b n: int): int
  requires { 0 <= n }
  requires { exists k. 0 <= k /\ a = fib k /\ b = fib (k+1) }
  ensures  { exists k. 0 <= k /\ a = fib k /\ b = fib (k+1) /\
                                              result = fib (k+n) }
  variant  { n }
= if n = 0 then a else aux b (a+b) (n-1)

let fib_rec (n: int): int
  requires { 0 <= n }
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= aux 0 1 n

(* fib_rec 5 = aux 0 1 5 = aux 1 1 4 = aux 1 2 3 =
               aux 2 3 2 = aux 3 5 1 = aux 5 8 0 = 5 *)
```

Instead of an existential we can use a ghost parameter:

```
let rec aux (a b n: int) (ghost k: int): int
  requires { 0 <= n }
  requires { 0 <= k /\ a = fib k /\ b = fib (k+1) }
  ensures  { result = fib (k+n) }
  variant  { n }
= if n = 0 then a else aux b (a+b) (n-1) (k+1)

let fib_rec (n: int): int
  requires { 0 <= n }
  ensures  { result = fib n }
= aux 0 1 n 0
```

The spirit of ghost code

Ghost code is used to facilitate specification and proof

$\Rightarrow$ the principle of non-interference:

We must be able to eliminate the ghost code
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  - if k is ghost, then $(k + 1)$ is ghost, too

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- ghost code cannot modify material data
  - if r is a material reference, then $r \leftarrow$ `ghost` k is forbidden
- ghost code cannot alter the control flow of material code
  - if c is ghost, then `if` c `then ...` and `while` c `do ...` are ghost

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  - if k is ghost, then $(k + 1)$ is ghost, too
- ghost code cannot modify material data
  - if r is a material reference, then `r ← ghost k` is forbidden
- ghost code cannot alter the control flow of material code
  - if c is ghost, then `if c then ...` and `while c do ...` are ghost
- ghost code cannot diverge
  - we can prove `while true do skip done ; assert false`

Can be declared ghost:

- function parameters

```
val aux (a b n: int) (ghost k: int): int
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- record fields and variant fields

```
type queue 'a = { head: list 'a;  (* get from head *)
                  tail: list 'a;  (* add to tail *)
            ghost elts: list 'a;  (* logical view *) }
invariant { elts = head ++ reverse tail }
```

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- local variables and functions

```
let ghost x = qu.elts in ...
let ghost rev_elts qu = qu.tail ++ reverse qu.head
```

Ghost code in WhyML

Can be declared ghost:

- function parameters

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val aux (a b n: int) (ghost k: int): int
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- local variables and functions

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let ghost x = qu.elts in ...
let ghost rev_elts qu = qu.tail ++ reverse qu.head
```

- program expressions

```
let x = ghost qu.elts in ...
```

How it works?

The material world and the ghost world are built from the same bricks.

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$$\Gamma \vdash e : \varsigma$$

ς — `int`, `bool`, `unit` (also: lists, arrays, etc.)

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ε — potential side effects

modified references	$r \leftarrow \ldots$, `let ref` $r = \ldots$ `in`
raised exceptions	`raise` E, `try` ... `with` E $\rightarrow$
divergence	unproved termination

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$\mathfrak{g}$ — is the expression material or ghost?

$\mathfrak{m}$ — is the expression's result material or ghost?

Who's ghost and who's not?

Any variable or reference is considered ghost

- if explicitly declared ghost: `let ghost` $v^{\mathfrak{g}}$ `= 6 * 6 in ...`
- if initialised with a ghost value: `let ref` $r^{\mathfrak{g}}$ `=` $v^{\mathfrak{g}}$ `+ 6 in ...`
- if declared inside a `ghost` block: `ghost (let` $x^{\mathfrak{g}}$ `= 42 in ...)`

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1. term t is ghost $\equiv$ t contains a ghost variable or reference

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   unless we pass a ghost value with E: `raise E v𝔤`

   unless E is expected to carry ghost values: `exception E (ghost int)`

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An expression e has a material effect iff

- e modifies a material reference
- e diverges ($=$ not proved to terminate)
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6. `try` e_1 `with` $E \rightarrow e_2$ / `try` e_1 `with` $E\ v \rightarrow e_2$ is ghost $\equiv$
   - e_1 is ghost
   - e_2 is ghost and raises an exception

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7. `if` t `then` e_1 `else` e_2 is ghost $\equiv$
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8. `while` t `do` e `done` is ghost $\equiv$ t or e is ghost

9. function call $f\ t_1 \ldots t_n$ is ghost $\equiv$
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- then the ghost status of every subexpression can be inferred

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Erasure $\lceil \cdot \rceil$ erases ghost data and turns ghost code into `skip`.

Theorem*: Erasure preserves the material part of program semantics.

$$\begin{array}{ccccccc} e \cdot \mu & \longrightarrow^{\star} & c \cdot \mu' & \qquad & e \cdot \mu & \Longrightarrow & \infty \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \lceil e \rceil \cdot \lceil \mu \rceil & \longrightarrow^{\star} & \lceil c \rceil \cdot \lceil \mu' \rceil & & \lceil e \rceil \cdot \lceil \mu \rceil & \Longrightarrow & \infty \end{array}$$

Lemma functions

General idea: a function $f\ \vec{x}$ `requires` P_f `ensures` Q_f that

- produces no results
- has no side effects
- terminates

provides a constructive proof of $\forall \vec{x}. P_f \rightarrow Q_f$

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$\Rightarrow$ a pure recursive function simulates a proof by induction

```
function rev_append (l r: list 'a): list 'a = match l with
  | Cons a ll -> rev_append ll (Cons a r)  |  Nil -> r end

let rec lemma length_rev_append (l r: list 'a)  variant {l}
  ensures { length (rev_append l r) = length l + length r }
=
  match l with Cons a ll -> length_rev_append ll (Cons a r)
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- by the postcondition of the recursive call:

  ```
  length (rev_append ll (Cons a r)) = length ll + length (Cons a r)
  ```

- by definition of `rev_append`:

  ```
  rev_append (Cons a ll) r = rev_append ll (Cons a r)
  ```

- by definition of `length`:

  ```
  length (Cons a ll) + length r = length ll + length (Cons a r)
  ```

9. Mutable data

Records with mutable fields

```
module Ref
  type ref 'a = { mutable contents : 'a } (* as in OCaml *)
  function (!) (r: ref 'a) : 'a = r.contents
  let ref (v: 'a) = { contents = v }
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end
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- can be passed between functions as arguments and return values
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  - `let r = ref 0 in while !r < 42 do r := !r + 1 done`
  - `val gr : ref int`

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  - `let ghost r = ref 42 in ... ghost (r := -!r) ...`

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- can hold ghost data
  - `let ghost r = ref 42 in ... ghost (r := -!r) ...`
- cannot be stored in recursive structures: no `list (ref 'a)`
- cannot be stored under abstract types: no `set (ref 'a)`

```
let double_incr (s1 s2: ref int): unit writes {s1,s2}
  ensures { !s1 = 1 + old !s1 /\ !s2 = 2 + old !s2 }
= s1 := 1 + !s1; s2 := 2 + !s2

let wrong () =
  let s = ref 0 in
  double_incr s s;   (* write/write alias *)
  assert { !s = 1 /\ !s = 2 }   (* in fact, !s = 3 *)
```

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```
val g : ref int

let set_from_g (r: ref int): unit writes {r}
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= r := !g + 1

let wrong () =
  set_from_g g;    (* read/write alias *)
  assert { !g = !g + 1 }    (* contradiction *)
```

WP vs. aliases

The standard WP rule for assignment:

$$\mathrm{WP}(x \leftarrow 42, Q[x, y, z]) = Q[42, y, z]$$

But if x and z are two names for the same reference

$$\mathrm{WP}(x \leftarrow 42, Q[x, y, z]) \quad \text{should be} \quad Q[42, y, 42]$$

Problem: Know, *statically*, when two values are aliased.

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Problem: Know, *statically*, when two values are aliased.

Solution: Tweak the type system and use inference (of course!)

WP with aliases

Every mutable type carries an *invisible identity token* — a region:

$$x : \texttt{ref}\ \rho\ \texttt{int} \qquad y : \texttt{ref}\ \pi\ \texttt{int} \qquad z : \texttt{ref}\ \rho\ \texttt{int}$$

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...fortunately: $\mathrm{WP}(\texttt{let}\ r = x\ \textit{or maybe}\ y\ \texttt{in}\ r \leftarrow 42,\ Q[x, y, z]) = ?$

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ML-style type inference reveals the identity of each subexpression

- formal parameters and global references are assumed to be separated

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ML-style type inference reveals the identity of each subexpression

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Revised WP rule for assignment: $\mathrm{WP}(x_\tau \leftarrow t, Q) = Q\sigma$

where σ replaces in Q each variable $y : \pi[\tau]$ with an updated value

- an alias of x can be stored inside a reference inside a record inside a tuple

Poor man's resizable array:

```
let resa = ref (Array.make 10 0) in
 (* resa : ref ρ (array ρ1 int) *)
```

Can we do more?

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Let's resize it:

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Change the type of `resa`? What about `if ... then resa := newa`?

Yes, we can!

Let everybody keep their type:

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 (* resa : ref ρ (array ρ1 int) *)
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`newa`, `olda` — the witnesses of the type system corruption

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newa, olda — the witnesses of the type system corruption

What do we do with undesirable witnesses? — A.G. Capone

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Type-changing expressions have a special effect:

$$\texttt{writes}\ \rho \cdot \texttt{resets}\ \rho_1, \rho_2$$

e_1 ; e_2 is well-typed $\Rightarrow$ in every free variable of e_2, every region reset by e_1 occurs under a region written by e_1

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Type-changing expressions have a special effect:

$$\texttt{writes}\ \rho \cdot \texttt{resets}\ \rho_1, \rho_2$$

e_1 ; e_2 is well-typed $\Rightarrow$ in every free variable of e_2,
every region reset by e_1 occurs under a region written by e_1

Thus: resa and its aliases survive, but olda and newa are invalidated.

e_1 `;` e_2 is well-typed $\Rightarrow$ in every free variable of e_2,
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The reset effect also expresses freshness:

> If we create a fresh mutable value and give it region ρ,
> we invalidate all existing variables whose type contains ρ.

Killer effect

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Effect union (for sequence or branching):

x_τ survives $\varepsilon_1 \sqcup \varepsilon_2 \quad \Leftrightarrow \quad x_\tau$ survives both ε_1 and ε_2.

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Effect union (for sequence or branching):

$$x_\tau \text{ survives } \varepsilon_1 \sqcup \varepsilon_2 \quad \Leftrightarrow \quad x_\tau \text{ survives both } \varepsilon_1 \text{ and } \varepsilon_2.$$

Thus:

- the reset regions of ε_1 and ε_2 are added together,
- the written regions of ε_i invalidated by ε_{2-i} are ignored.

To sum it all up

The standard WP calculus requires the absence of aliases!

- at least for modified values
- WHY3 relaxes this restriction using static control of aliases

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- `val set_from_g (r: ref int): unit writes {r} reads {g}`
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- otherwise the static control of aliases does not have enough information

For programs with arbitrary pointers we need more sophisticated tools:

- memory models (for example, “address-to-value” arrays)
- handle aliases in the VC: separation logic, dynamic frames, etc.

Aliasing restrictions in WhyML

⇒ certain structures cannot be implemented

Still, we can specify them and verify the client code

```
type array 'a = private { mutable ghost elts: map int 'a;
                                            length: int }
  invariant { 0 <= length }
```

- all access is done via abstract functions (private type)
- the type invariant is expressed as an axiom
  - but can be temporarily broken inside a program function

```
type array 'a = private { mutable ghost elts: map int 'a;
                                         length: int }
  invariant { 0 <= length }

val ([]) (a: array 'a) (i: int): 'a
  requires { 0 <= i < a.length }
  ensures  { result = a.elts[i] }

val ([]<-) (a: array 'a) (i: int) (v: 'a): unit
  requires { 0 <= i < a.length }
  writes   { a }
  ensures  { a.elts = (old a.elts)[i <- v] }

function get (a: array 'a) (i: int): 'a = a.elts[i]
```

- the immutable fields are preserved — implicit postcondition
- the logical function `get` has no precondition
  - its result outside of the array bounds is undefined

10. Modular programming considered useful

Declarations

- types
  - abstract: `type t`
  - synonym: `type t = list int`
  - variant: `type list 'a = Nil | Cons 'a (list 'a)`
- functions / predicates
  - uninterpreted: `function f int: int`
  - defined: `predicate non_empty (l: list 'a) = l <> Nil`
  - inductive: `inductive path t (list t) t = ...`
- axioms / lemmas / goals
  - `goal G: forall x: int, x >= 0 -> x*x >= 0`
- program functions
  - abstract: `val ([]) (a: array 'a) (i: int): 'a`
  - defined: `let mergesort (a: array elt): unit = ...`
- exceptions
  - `exception Found int`

Specification language of WhyML

- programs and specifications use the same data types
- `match-with-end`, `if-then-else`, `let-in`
  are accepted both in terms and formulas
- functions et predicates can be defined recursively:

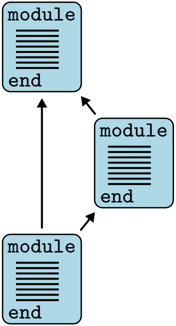
```
predicate mem (x: 'a) (l: list 'a) = match l with
  Cons y r -> x = y \/ mem x r | Nil -> false end
```

  no `variant`s, Why3 requires structural decrease
- inductive predicates (useful for transitive closures):

```
inductive sorted (l: list int) =
  | SortedNil: sorted Nil
  | SortedOne: forall x: int. sorted (Cons x Nil)
  | SortedTwo: forall x y: int, l: list int.
      x <= y -> sorted (Cons y l) ->
        sorted (Cons x (Cons y l))
```

Declarations are organized in modules

- purely logical modules are called theories

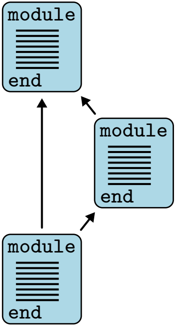


Declarations are organized in modules

- purely logical modules are called theories

A module M_1 can be

- used (`use`) in a module M_2
  - symbols of M_1 are shared
  - axioms of M_1 remain axioms
  - lemmas of M_1 become axioms
  - goals of M_1 are ignored



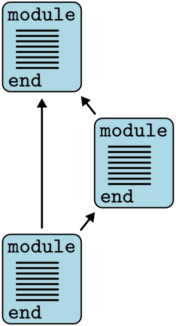
Modules

Declarations are organized in modules

- purely logical modules are called theories

A module M_1 can be

- used (`use`) in a module M_2
- cloned (`clone`) in a module M_2
  - declarations of M_1 are copied or instantiated
  - axioms of M_1 remain axioms or become lemmas
  - lemmas of M_1 become axioms
  - goals of M_1 are ignored



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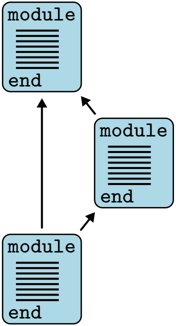
- an abstract type with a defined type
- an uninterpreted function with a defined function
- a `val` with a `let`



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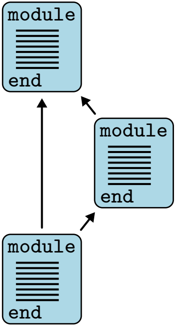
One missing piece coming soon:

- instantiate a used module with another module



Exercises

`http://why3.lri.fr/ejcp-2022`